Int. J. Solids Structures Vol. 29, No. 23, pp. 2921–2932, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

A GENERAL STOCHASTIC DYNAMIC MODEL OF CONTINUUM DAMAGE MECHANICS

C. W. Woo and D. L. Li
Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong

(*Received* 18 *July* 1991; *in revised form* 28 *April* 1992)

Abstract—A stochastic dynamic probabilistic model of continuum damage mechanics is generalized from which various damage processes could be described in the same mathematical scheme. The approach makes use of the deterministic damaged constitutive equations and introduces the randomized damage variable into the thermodynamic potential and the potential of dissipation in place of the deterministic ones. An Itô (or Stratonovich) stochastic differential equation is derived, based on the concept of an abstract dynamic system to describe the practical system under consideration. A solution as a diffusion process can be obtained in the light of the theory of the stochastic differential equation. The proposed model can be used to describe the state and stochastic evolution of various damage processes in relation to the irreversible rearrangements of micro structures of a solid. From the proposed model, both the stochastic characteristic of damage and the deterministic properties of damage embodied in the deterministic theory, such as the non-linearity of damage with regard to time or number of cycles and the effect of stress triaxiality on damage, can be modelled.

1. INTRODUCTION

Damage, as a physical phenomenon representing the formation of microcavitation and/or microcracking, exists unavoidably in engineering material and structure and evolves with some parameters, such as time, loading and environments. When damage reaches a critical value of the material under consideration, the failure of the component or structure will happen. In the classical continuum mechanics of solids, the constitutive equations of material in which the relationship of stress and strain or deformation are set up, are based on the general principles of thermodynamics and on the assumption of ideal material, i.e. no damage exists in the material. Fracture mechanics suggests an approach for damage represented by the ideal or regular crack with definite geometry and location. Fracture mechanics incorporating with the stress–strain analysis of components based on the classical continuum mechanics in an uncoupled manner provides an analytical procedure for damage assessment, and has been extensively used in engineering practice. However, it would be unsuitable before the appearance of macrocrack and difficult for some complex engineering structures in which the geometry and location of crack could not be determined precisely.

Continuum damage mechanics (CDM), as a phenomenological theory, seeks to describe the state and evolution of irreversible microstructural alterations or damage of material by considering systematically the effects of damage on macroscopic mechanical properties of material such as the strength and stiffness. The general framework of CDM is established on the thermodynamic theory of irreversible processes after identifying a proper set of internal variables characterizing the irreversible microscopic occurrences together with their conjugated generalized forces. The construction of damage constitutive equations in CDM is based on two potentials: the thermodynamic potential which is used to obtain the state laws of the non-dissipative phenomena and the definition of variables; and the dissipative potential which is used to construct the laws of evolution of the dissipative variables and processes. The result is a set of constitutive equations for all the variables introduced. The coupling among damage, stress and strain or deformation is automatically obtained through the damaged constitutive equations.

Since damage is by nature related to some microscopic mechanisms and is quite sensitive to certain environmental effects which vary in a stochastic manner, the stochastic characteristic in the constitutive equations or the potential functions of the material after the introduction of the damage variable will be unavoidable. This inspires us to develop the dynamic probabilistic theory of CDM. On the other hand, there are several additional facts that strongly support such a need. One is the lack of an exact description of the underlying damage process as arbitrariness in the mathematical description of the thermodynamic potential and the potential of dissipation for CDM. Another is the variation in fatigue performance and the discrepancy between observation and prediction. In principle, the problem could be approached by statistical mechanics, micromechanics and probabilistic mechanics of discrete media on the foundation of the microscopic theory. So far, however, the microscopic theory is still far away from the application in the engineering structures. From the practical point of view, the phenomenological approach based on the macroscopic mechanical properties of the material such as the strength and stiffness may be more suitable. In this respect, the CDM has become an appropriate choice, and it is therefore pertinent to incorporate a probabilistic model in the CDM using a randomized damage variable.

One approach to incorporate a stochastic probabilistic model is to deal with the deterministic constitutive equations and to introduce the randomized damage variable in place of the conventional damage variable, and at the same time to add a stochastic fluctuation process to describe the variability. In this approach, the evolution of the state of the damage as a function of some parameters (time without loss of generality) is described. Another approach is to assume an evolutionary probabilistic model to describe the damage process from the beginning. In this approach the evolution of the probability distribution of the damage state and the evolution as a function of time is then considered. Both approaches require the introduction of a probabilistic model based on certain assumptions or hypotheses. In the investigation of numerous physical and engineering problems, the hypothesis that is used most extensively is Markovian [see Howard (1971)]. The Markov property, based on the principle that the "future" is independent of the "past" when the "present" is known, is the causality principle of classical physics carried over to stochastic dynamic systems. On the other hand, the hypothesis makes it possible to use a large variety of mathematical schemes elaborated in the theory of Markov stochastic processes, and could lead to interesting results.

2. DETERMINISTIC THEORY OF CONTINUUM DAMAGE MECHANICS

Continuum Damage Mechanics (CDM) supported by the thermodynamic theory of irreversible processes has been developed continuously since the pioneering works by Kachanov (1958) and Rabotnov (1963). At present, it has evolved as a practical tool to take into account the various damaging processes in solid materials and structures at a macroscopic continuum level.

Before developing a general stochastic dynamic model, a brief review of several concerned concepts and definitions is presented here.

General framework of CDM

Continuum damage mechanics, as a phenomenological theory, is established from the general principles of thermodynamics. A state variable, as an internal variable, representing the irreversible microstructural rearrangements manifested in the modes of diffuse flow, change of phase, change in porosity, crystalline slip, twinning and loss of cohesion along the grain boundaries or cleavage planes, is introduced into the constitutive equations of solids. Based on the assumptions (a) that the response of a system depends only on the current state of the system under consideration, and (b) that a thermodynamically irreversible process can be approximated by a sequence of constrained equilibrium states, the application of the internal variable method leads to a sufficiently close approximation of a given non-equilibrium state by a constrained equilibrium state (Kestin and Bataille, 1977).

Damage variable

For the solution over a wide range of responses, damage as an internal variable should have not only a real physical meaning, but should also be able to reflect the dominant mechanism of irreversible rearrangements and dissipative processes of the system under consideration. Since damage is a comprehensive manifestation of a host of microscopic responses which is related to a complex thermodynamic process, it would not be easy to identify such a parameter. Therefore a lot of effort has been made using CDM to identify a suitable damage variable with the physical requirements described above and with the mathematical accessibility. The various scalar, vectorial and tensorial definitions with the mechanical, geometrical and physical interpretations can be found in a number of reviews by several distinguished investigators [see, for example, Chaboche (1981, 1988a,b), Kachanov (1986), Krajcinovic (1984), Krajcinovic and Lemaître (1987), Lemaître (1984, 1986, 1987), Lemaître *et al.* (1987) and Murakami (1987)].

Thermodynamic variables

In principle, it is always possible to select a proper set of internal variables which will lead to a sufficiently close approximation of a given non-equilibrium state by a constrained equilibrium state through the constitutive equations. A set of internal variables developed by Lemaître (1987) is limited to isotropic hardening for plasticity and to the isotropic damage theory. For more general cases, the set of internal variables, together with their associated variables or conjugate forces, is listed in Table 1.

In general, the variables defined in Table 1 are suitable for most strain hardening models and damage models including both isotropic and anisotropic models. For the mechanical response, in the usual sense, isothermal conditions are considered so that the effect of temperature enters the constitutive equations only through material parameters.

Essential postulates of CDM

To establish the damaged constitutive equations, it is necessary to relate the damage to other internal variables by some physical hypotheses. The postulate which is used extensively in CDM is the hypothesis of strain equivalence. This hypothesis is basically empirical in nature, and was tacitly adopted from the beginning of CDM by Kachanov (1958) and later was phrased as an essential principle by Lemaître and Chaboche (1978) [see also Lemaître (1971) and Chaboche (1977)]. It states: Any strain constitutive equation for a damaged material is derived from the same potentials as for a virgin material except that all the stress variables are replaced by effective stresses. It relates to the concept of effective stress $\tilde{\sigma}$ defined by

$$\tilde{\boldsymbol{\sigma}} = \mathbf{M}(\mathbf{D}) : \boldsymbol{\sigma}, \tag{1}$$

where $\mathbf{M}(\mathbf{D})$ is a fourth rank symmetric tensor, named as the "damage effect tensor" and the symbol ":" means the tensorial product contracted on two indices. In general, $\mathbf{M}(\mathbf{D})$ has 21 independent elements. One possible formulation was proposed by Cordebois and Sidoroff (1982) and subsequently by Chow and Wang (1987a,b) in the principal coordinate system which has only six elements that are non-zero. While damage is considered as isotropy, (1) is reduced to a scalar form:

$$\tilde{\sigma} = \sigma/(1-D) \tag{2}$$

which relates the stress to the area supporting the load effectively (Rabotnov, 1969; Janson, 1977).

Table 1. Thermodynamic variables

State variables		Associated variables	
Strain tensor	$\boldsymbol{\varepsilon}$	Cauchy stress tensor	$\boldsymbol{\sigma}$
Temperature	θ	Entropy	s
Elastic strain tensor	$\boldsymbol{\varepsilon}^e$	Cauchy stress tensor	$\boldsymbol{\sigma}$
Plastic strain tensor	$\boldsymbol{\varepsilon}^p$	Cauchy stress tensor	$-\boldsymbol{\sigma}$
Accumulative plastic strain	p	Strain hardening threshold	$\mathbb{R}$
Damage tensor	$\mathbf{D}$	Damage energy release rate	$\mathbf{Y}$
Overall damage	w	Damage strengthening threshold	$\mathbb{B}$

The hypothesis of strain equivalence, associated with the concept of effective stress, yields a strain-based formulation of damaged constitutive equations. However, a stress-based characterization of the material response is sometimes necessitated as most elasto–plastic models in the theory of plasticity are formulated in stress space. Thus, a hypothesis of stress equivalence, similar to the strain equivalent postulate except that strain is replaced by stress, is proposed by Simo and Ju (1987a,b). Since anisotropic damage is defined as a second order tensor [see Lu and Chow (1990)], the postulates of both strain equivalence and stress equivalence lead to an unsymmetrical stiffness or compliance matrix and hence may be thermodynamically inadmissible. To overcome this, the hypothesis of elastic energy equivalence (Sidoroff, 1981) and the hypothesis of stress working equivalence (Lu and Chow, 1990) are proposed.

Thermodynamic potential

Based on the thermodynamic theory, under the isothermal condition the state of the damaged material is defined through a thermodynamic potential per unit mass expressed by

$$\psi(\mathbf{x}, t) = \psi(\varepsilon^{e}, \varepsilon^{p}, p, \mathbf{D}, w, \mathbf{x}, t), \tag{3}$$

where $\mathbf{x}$ denotes a material point, t is time and others are defined in Table 1. By using a partial Legendre–Fenchel transformation, the dual potential of (3), the complementary energy per unit mass, can be obtained as

$$\rho\psi^{*}(\mathbf{x}, t) = \boldsymbol{\sigma} : \varepsilon^{e} - \rho\psi(\mathbf{x}, t), \tag{4}$$

where ρ is the density of the matter. Upon substitution of (3) and (4) into the Clausius–Duhem inequality one obtains

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\varepsilon^{e}};\ \varepsilon^{e} = \rho\frac{\partial\psi^{*}}{\partial\boldsymbol{\sigma}};\ \mathbb{R} = -\rho\frac{\partial\psi}{\partial p};\ \mathbb{B} = \rho\frac{\partial\psi}{\partial w};\ \mathbf{Y} = -\rho\frac{\partial\psi}{\partial\mathbf{D}}. \tag{5}$$

From the normality rule, (5), and thermodynamic potential, (3), the state laws of the materials with damage can be derived.

Potential of dissipation

The evolution of damage, as a dissipative process of irreversible rearrangement of microstructures, can be described by flux variables and their conjugate thermodynamic forces. Table 2 lists a set of the flux variables and their conjugate dual variables where the symbol "·" on top of a letter denotes the derivative with respect to time, where $\mathbf{q}$ is the outward heat flux vector and θ is temperature.

The complementary kinetic laws of damage evolution can be derived from the dissipative potential which is postulated as a convex non-negative scalar function of all flux variables and the state variables acting as parameters, i.e.

Table 2. Flux variables and their conjugate dual variables

Flux variables	Dual variables
$\dot{\varepsilon}^{p}$	$\boldsymbol{\sigma}$
$-\dot{p}$	$\mathbb{R}$
$-\dot{\mathbf{D}}$	$\mathbf{Y}$
$-\dot{w}$	$\mathbb{B}$
$\mathbf{q}/\theta$	$\mathbf{grad}\ \theta$

$$\phi(\mathbf{x},t) = \phi(\dot{\varepsilon}^{p},\dot{p},\dot{\mathbf{D}},\dot{w},\dot{\mathbf{q}}/\theta;\ \varepsilon^{e},\varepsilon^{p},p,\mathbf{D},w,\theta,\mathbf{x},t) \tag{6}$$

and the dual function can be obtained by partial or total Legendre–Fenchel transform as

$$\phi^{*}(\mathbf{x},t) = \phi^{*}(\sigma,\mathbb{R},\mathbf{Y},\mathbb{B},\mathbf{grad}\,\theta;\ \varepsilon^{e},\varepsilon^{p},p,\mathbf{D},w,\theta,\mathbf{x},t). \tag{7}$$

The dissipation rate is expressed by the Clausius–Duhem inequality which must be positive to satisfy the second principle of thermodynamics. Together with the state laws the generalized normality rules can be obtained as

$$\dot{\varepsilon}^{p} = \frac{\partial\phi^{*}}{\partial\sigma};\ \dot{p} = -\frac{\partial\phi^{*}}{\partial\mathbb{R}};\ \dot{\mathbf{D}} = -\frac{\partial\phi^{*}}{\partial\mathbf{Y}};\ \dot{w} = -\frac{\partial\phi^{*}}{\partial\mathbb{B}};\ \frac{\mathbf{q}}{\theta} = \frac{-\partial\phi^{*}}{\partial\,\mathbf{grad}\,\theta}. \tag{8}$$

The most important step in the determination of the damaged constitutive equations is the selection of analytical expressions for the potential of dissipation used to define the kinetic laws.

3. STOCHASTIC DYNAMIC PROBABILISTIC MODELLING

To describe the damage of the structured solids using the stochastic dynamic theory, we conceive an abstract dynamic system representing the state and evolution of the damage material under consideration. The system has the properties as follows:

Property 1. Structured solids under consideration in the system are described as a continuous system which means that the molecular structure of matter will be disregarded. Under the assumption, all variables defined in CDM, such as the variables in Table 1 and the theory accompanied by the relevant conception developed by CDM are valid in the system.

Property 2. In the process of damage evolution, the system's continuity and homogeneity remain the same as the original. In light of this assumption, the system is still described as continuum when damage in some point or element reaches some critical value in which the stiffness and strength have vanished.

Property 3. Damage variable $\mathbf{D}$, as a continuum or discrete state variable, varies in a random manner in a random field with a continuum or discrete parameter t (time, without loss of generality), designated as $\mathbf{D}_t$, and has the random initial value $\mathbf{D}_{t_0} = \mathbf{D}_0$.

Property 4. The stochastic characteristic of the state and evolution of damage in the system is represented by a small stochastic fluctuation of a white Gaussian noise ξ_t, for $-\infty < t < \infty$, with mean $E\xi_t = 0$ and a constant spectral density on the entire real axis.

In this paper, for the sake of simplicity, we confine ourselves to isotropic damage although no restriction on anisotropy is imposed in the abstract dynamic system. Therefore, the damage tensor $\mathbf{D}$ and damage energy release rate $\mathbf{Y}$ degenerated to the scale form D and Y, respectively.

To develop the stochastic dynamic modelling starting from the deterministic constitutive equations, we need to extend the relevant concepts and the expressions in CDM to proper random fields, based on the abstract dynamic system conceived.

Let $\{S_t;\ t \in [t_0, T]\}$, denoted simply as S_t, represent a stochastic process whose state space is a d-dimensional Euclidean space R^{d} (for $d \geqslant 1$) and whose index set is defined in an interval $[t_0, T]$ of the real axis R^{1} which can in general be $(-\infty, +\infty)$. For our purposes, it will be sufficient in all cases to assume

$$[t_0, T] \subset [0, \infty) = R^+.$$

We shall always assume that the state space R^d is endowed with the sigma-algebra $\mathfrak{B}^d$ of Borel sets and assume that all processes to be discussed below are defined on a certain probability space $(\Omega, \mathfrak{U}, P)$ $\Omega = (R^d)^{[t_0, T]}$ is the space of all R^d-valued functions defined on the interval $[t_0, T]$; $\mathfrak{U} = (\mathfrak{B}^d)^{[t_0, T]}$ is the product sigma-algebra generated by the Borel sets in R^d; and P is the probability defined by the finite-dimensional distributions of the process S_t on $(\Omega, \mathfrak{U})$.

In our stochastic dynamic system, besides the evolution of the damage, all other variables and potentials after introducing the randomized damage variable will contain the deterministic relationships similar to those in CDM.

With the assumption of continuum, the rate of damage evolution in the material can be represented mathematically by a differential equation of the form

$$\dot{D}_t = f(t, D_t, X_t), \quad t \geqslant t_0, \quad D_{t_0} = D_0, \tag{9}$$

in which D_t is the randomized state variable of damage defined in the d-dimensional Euclidean space R^d, and X_t is the random disturbance process defined in the m-dimensional Euclidean space R^m. In this paper, we confine ourselves to $d = m = 1$ cases. $\dot{D}_t$ denotes the derivative of D_t to parameter t with the interval $[t_0, T]$ in which $t_0 \geqslant 0$ and $T < \infty$. In general, $f(t, D_t, X_t)$ could be non-linear in X_t.

In light of Property 4 described above, the disturbance process in the system is represented by a small fluctuation of a white Gaussian noise ξ_t. So the random disturbance process is independent with D_t. Therefore, the evolution of damage in our system could be considered as linear in X_t, so that (9) has the form

$$\dot{D}_t = f(t, D_t) + G(t, D_t)X_t, \quad t \geqslant t_0, \quad D_{t_0} = D_0. \tag{10}$$

In this model, X_t could be described by any independent random process with small fluctuation. If the fluctuation is a stochastic process with sufficiently smooth or continuous sample functions, (10) can be considered as an ordinary differential equation for the sample functions of the state of the system. When X_t does not depend at all on chance but is equal to a fixed function, especially $X_t \equiv 0$, (10) degenerates into a deterministic equation.

For no other reason than that of the physical phenomenon as well as the simplicity in mathematics, we introduce the white noise ξ_t as the prototype of a delta-correlated Gaussian noise process representing the fluctuating process. Thus our model has the form

$$\dot{D}_t = f(t, D_t) + G(t, D_t)\xi_t, \quad t \geqslant t_0, \quad D_{t_0} = D_0, \tag{11}$$

for which a precise mathematical theory, by virtue of the stochastic integral, has been developed [see Arnold (1974)]. The solution is a Markov process and the efficient methods exist for the mathematical analysis of this type of process. However, it has the disadvantage that its sample functions are not smooth functions [see Howard (1971)]. This is typical of the Markov processes because the Markov property formulated in a negative way states that: for a known "present" it is forbidden to transmit information from the "past" into the "future".

Even though the white noise ξ_t is not a usual stochastic process, nonetheless the indefinite integral of ξ_t can be identified by an R^1-valued Wiener process W_t, i.e.

$$W_t = \int_0^t \xi_t \, dt,$$

or, in shorter symbolic notation when we consider both processes as generalized stochastic processes, as

$$dW_t = \xi_t \, dt.$$

Therefore (11) could be re-written as

$$dD_t = f(t, D_t)\,dt + G(t, D_t)\,dW_t, \quad t_0 \leqslant t \leqslant T < \infty, \quad D_{t_0} = D_0. \tag{12}$$

We have not yet considered $f(t, D_t)$ and $G(t, D_t)$ in (10) even though (12) has been developed. Obviously, $f(t, D_t)$ and $G(t, D_t)$ will depend on the constitutive equations of material since our approach is to start with the deterministic approach accompanied by an abstract dynamic system. Two assumptions are important in developing the model. Assumption 1 is that $f(t, D_t)$ in (12) should be identical to the deterministic constitutive equations of material under consideration. Although it is unnecessary in developing a stochastic dynamic model because of the noise-induced shift, it is reasonable to compound the shift into the deterministic constitutive equations of material from the beginning. In fact, the definition of the damage variable D in CDM has been explained as a statistical mean value. This means that the deterministic constitutive equations developed in CDM could contain the noise-induced shift. Assumption 2 is that the intensity of fluctuation is directly proportional to the mean rate of damage evolution described by the deterministic constitutive equations. This assumption is made because of the consideration of physics rather than one of mathematics.

From these two assumptions and the generalized normalized rules for the rate of increase of internal variables, (8), a general stochastic dynamic model for damage is proposed in the form

$$\dot{D}_t = -\frac{\partial \phi_t^*}{\partial Y_t} + \zeta \frac{\partial \phi_t^*}{\partial Y_t} \xi_t, \quad t_0 \leqslant t \leqslant T < \infty, \quad D_{t_0} = D_0, \tag{13}$$

or

$$dD_t = -\frac{\partial \phi_t^*}{\partial Y_t}\,dt + \zeta \frac{\partial \phi_t^*}{\partial Y_t}\,dW_t, \quad t_0 \leqslant t \leqslant T < \infty, \quad D_{t_0} = D_0, \tag{14}$$

where ζ is a proportional constant. It should be noted that subscript t have been added to ϕ^* and Y to distinguish between deterministic and stochastic ones.

In accordance with this model, the stochastic dynamic laws could be derived from the generalized potential function of dissipation. For example, from the potential function proposed by Lemaître (1987) a stochastic damage constitutive equation can be obtained as

$$dD_t = \frac{Y_t(\dot{p} + \dot{\pi})}{S_0(1 - D_t)^{\alpha_0}}\,dt + \frac{Y_t(\dot{p} + \dot{\pi})}{S_1(1 - D_t)^{\alpha_0}}\,dW_t, \quad t_0 \leqslant t \leqslant T < \infty, \quad D_{t_0} = D_0, \tag{15}$$

where S_0 and α_0 are two coefficients of the material characteristic whereas $S_1 = S_0/\zeta$ is a stochastic characteristic of the material. In Lemaître theory, π represents the rate of the accumulated microplastic strain and

$$Y = \left[\frac{(1+\upsilon)\langle\sigma\rangle : \langle\sigma\rangle - \upsilon\langle \mathrm{tr}(\sigma)\rangle^2}{2E(1-D)^2} + h\frac{(1+\upsilon)\langle-\sigma\rangle : \langle-\sigma\rangle - \upsilon\langle-\mathrm{tr}(\sigma)\rangle^2}{2E(1-Dh)^2}\right]\frac{-1}{(1-k\pi^{1/m})},$$

where υ and E are elasticity coefficients varying with temperature; h is a closure coefficient which characterizes the closure of the microcracks and microcavities; k and m are microplasticity coefficients. In the formulation, the Cauchy stress tensor was divided into a "positive" part $\langle\sigma\rangle$ and a "negative" part $\langle-\sigma\rangle$.

Equation (14) could be used to describe the stochastic evolution of various types of damage to the material and structure, such as ductile, fatigue, creep and so on. The admissibility of the model proposed will be supported by both the reasonable description of the physical phenomenon and the accessibility of the mathematics.

4. PROPERTIES AND SOLUTION OF THE STOCHASTIC MODEL

The proposed model, (14), is based on an abstract dynamic system. In this system, the stochastic fluctuation is described by a white Gaussian noise ξ_t, for $-\infty < t < \infty$, with zero mean and a constant spectral density on the entire real axis. This is, as it is known, a non-physical but abstract and useful idealization. The infinite integral of ξ_t can be identified with a Wiener process with mean $EW_t = 0$ and with covariance $EW_tW_s = \min(t, s)$. Therefore, (14) could be interpreted as an abbreviation for the integral equation

$$D_t = D_0 - \int_{t_0}^{t} \frac{\partial \phi_t^*}{\partial Y_t}\,\mathrm{d}t + \int_{t_0}^{t} \zeta \frac{\partial \phi_t^*}{\partial Y_t}\,\mathrm{d}W_t, \quad t_0 \leqslant t \leqslant T < \infty, \quad D_{t_0} = D_0. \tag{16}$$

Since the sample functions of W_t are, with probability 1, continuous though not of bounded variation in any interval, the second integral in (16) cannot be regarded in general as an ordinary Riemann–Stieltjes integral with respect to the sample functions of W_t. This is because in the attempt to evaluate the integral

$$\int_{t_0}^{t} W_t\,\mathrm{d}W_t$$

as the limiting value of the approximating sums

$$S_n = \sum_{i=1}^{n} W_{\tau_i}(W_{t_i} - W_{t_{i-1}}), \quad t_0 \leqslant t \leqslant \cdots \leqslant t_n = t, \quad t_{i-1} \leqslant \tau_i \leqslant t_i,$$

the result depends very much on the choice of the intermediate point τ_i. A different choice of τ_i will provide a different interpretation of the stochastic differential equation. Itô's choice of $\tau_i = t_{i-1}$ leads to an unsymmetrical integral

$$\int_{t_0}^{t} W_t\,\mathrm{d}W_t = \underset{\delta n \to 0}{\text{qm-lim}}\, S_n = (W_t^2 - W_{t_0}^2)/2 - (t - t_0)/2$$

with respect to the variable t since the increments $\mathrm{d}W_t$ point into the future (Itô, 1951), in which qm-lim denotes quadratic mean or mean square limit, and $\delta n = \max(t_i - t_{i-1})$. It results in the discrepancy between the stochastic differential equation and the ordinary differential equation. However, it is just this lack of symmetry that leads to the simple formulae for the first two moments of the integral and to the Martingale property [see Arnold (1974)]. Furthermore, from Itô's theorem a diffusion process can be obtained as the solution of the stochastic differential equation (12). The intuitive significance of the coefficients $f(t, D_t)$ and $G(t, D_t)$ is explained by regarding $f(t, D_t)$ as the drift vector and

$$B(t, D_t) = G(t, D_t)G'(t, D_t), \tag{17}$$

as the diffusion matrix of that process, in which $G'(t, D_t)$ is the transpose of the matrix $G(t, D_t)$. Therefore, in accordance with the definition of the diffusion process the first two moments are

$$\lim_{t \downarrow \tau} \frac{1}{t-\tau} \int_{|y-x| \leqslant \varepsilon} (y-x) P(\tau, x, t, \mathrm{d}y) = f(\tau, x) \tag{18}$$

and

$$\lim_{t \downarrow \tau} \frac{1}{t-\tau} \int_{|y-x| \leq \varepsilon} (y-x)(y-x)' P(\tau, x, t, \mathrm{d}y) = B(\tau, x), \tag{19}$$

where $\varepsilon > 0$, $\tau \in [t_0, T]$, $x \in R^d$ and $y \in R^d$.

Another interpretation is based on the definition of Stratonovich's time-symmetric stochastic integral (Stratonovich, 1966). In accordance with Stratonovich's definition, the unsymmetrical part in the integral

$$\int_{t_0}^{t} W_t \, \mathrm{d}W_t = \text{qm-}\lim_{\delta n \to 0} \sum_{i=1}^{n} \frac{W_{t_{i-1}} + W_{t_i}}{2} (W_{t_i} - W_{t_{i-1}}) = (W_t^2 - W_{t_0}^2)/2$$

vanishes. Therefore, the solution of (16) can be obtained by formal integration by parts.

Obviously, the two different mathematical definitions will lead to different solutions for our model. These discrepancies arise not from the errors in the mathematical calculation but from the general discontinuity of the relationship between differential equations for stochastic processes and their solutions. Though Stratonovich's definition matches the consistency between the stochastic differential equation and ordinary differential equation, it is, however, difficult to judge which of the two is the correct definition because there is no reason why the definition of the stochastic differential equation should be consistent with that of the ordinary differential equation. In fact, Itô's equation should be equivalent to Stratonovich's equation through a mathematical transform as

$$\mathrm{d}D_t = \left(f(t, D_t) + \frac{\partial G(t, D_t)}{2\partial D_t} G(t, D_t) \right) \mathrm{d}t + G(t, D_t) \, \mathrm{d}W_t. \tag{20}$$

In view of the above reasons, we adopt Itô's interpretation, from which the drift coefficient $f(t, D_t)$ identifies with the deterministic constitutive equations in CDM. This is one of the reasons for Assumption 1 described in Section 3 to be taken.

There are in general two approaches for the solution of our model: One is directly concerned with the random damage variable D_t and its derivative. This belongs to the probabilistic or direct method. Another is the so-called analytical or indirect probability method which does not deal with the timewise development of the state D_t, but, for example, with the timewise development of transition probabilities $P(D_t \in B | D_\tau = x)$. In accordance with the theory of the stochastic differential equation, the solution of (14) is a diffusion process. For the diffusion process, there exist effective solutions for the analytical probability method. For example, we can obtain the transition density $P_d(\tau, D_\tau, t, D_t)$ from a fundamental solution of the Kolmogorov's forward or the Fokker–Planck equation:

$$\frac{\partial p_d}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial D_{t_i}} (f_i(t, D_t) p_d) - \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial D_{t_i} \partial D_{t_j}} (b_{ij}(t, D_t) p_d) = 0, \tag{21}$$

where $b_{ij}(t, D_t)$ are the elements of the diffusion matrix $B(t, D_t)$. In this paper, we do not describe the solution process since it can be found in relevant references [see Arnold (1974)]. Instead, we briefly discuss the existence and uniqueness of a solution of the proposed model.

According to the theory of the stochastic differential equation, to ensure the existence and uniqueness of a solution of the equation

$$\mathrm{d}D_t = f(t, D_t) + G(t, D_t) \, \mathrm{d}W_t, \quad t_0 \leq t \leq T < \infty, \quad D_{t_0} = D_0,$$

where W_t is an R^m-valued Wiener process and D_0 is a random variable independent of $W_t - W_{t_0}$ for $t \geq t_0$. The R^d-valued function $f(t, D_t)$ and the ($d \times m$ matrix)-valued function $G(t, D_t)$ should be defined and measurable on $[t_0, T] \times R^d$ and satisfy the following properties. There exists a constant $K > 0$ such that:

(a) for all $t \in [t_0, T]$, $x \in R^d$ and $y \in R^d$,

$$|f(t,x)-f(t,y)|+|G(t,x)-G(t,y)| \leqslant K|x-y|;$$

(b) for all $t \in [t_0, T]$ and $x \in R^d$,

$$|f(t,x)|^2+|G(t,x)|^2 \leqslant K^2(1+|x|^2).$$

In our model (14),

$$G(t,D_t) = \zeta F(t,D_t) = \zeta \frac{\partial \phi^*}{\partial Y},$$

where ζ is a bounded constant characterizing the intensity of random fluctuation. Therefore, the conditions of existence and uniqueness of a solution for the stochastic differential equation (14) will be the same for an ordinary differential equation. In other words, our model has a unique R^d-valued solution D_t in the index set $[t_0, T] \subset [0, \infty)$ of the stochastic process, and continuous with probability 1 so long as a solution of the deterministic constitutive equations in CDM satisfies the conditions of existence and uniqueness.

5. DISCUSSION

Although the definition of damage in CDM is based on the concept of the statistical mean, the evolution process is sensitive to environmental effects which often vary in a random manner and usually cannot be controlled or measured in practice. The need for incorporating the stochastic characteristic in the damage variable D is therefore evident. In addition to the physical consideration, the development of probabilistic CDM can help to solve complex field problems in areas of engineering such as aerospace and offshore since exact analysis using the deterministic theory of CDM is extremely difficult in these areas.

The proposed model is intended to bridge the gap between CDM and the probabilistic CDM. The analysis reduces to a stochastic differential equation for which there exists a powerful analytical means at our disposal. In accordance with the theory of the stochastic differential equation, the solution process of our model is a diffusion process with the drift coefficient $f(t, D_t)$ and the diffusion matrix $B(t, D_t)$. This is a special case of the Markov process with continuous sample functions which at first serve as probability-theoretic models of physical diffusion phenomena, but later have been proven to be suitable for stochastic modelling of a wide variety of physical phenomena. If we first assume that damage evolution can be described by a diffusion process, the first two moments of the increment $D_t - D_\tau$ under the condition $D_\tau = x$ as $t \downarrow \tau$ become

$$E_{\tau,x}(D_t - D_\tau) = f(\tau, x)(t-\tau)+O(t-\tau) \tag{22}$$

and

$$E_{\tau,x}(D_t - D_\tau)(D_t - D_\tau)' = B(\tau, x)(t-\tau)+O(t-\tau). \tag{23}$$

Therefore, $f(\tau, x)$ denotes the mean rate of the random damage evolution described by D_t under the assumption $D_\tau = x$. $B(\tau, x)$ is a measure of the local magnitude of the fluctuation of $D_t - D_\tau$ about the mean value. These are, in physics, consistent with Assumptions 1 and 2 described in Section 3. This is also one of the reasons for us to take these assumptions.

6. CONCLUSIONS

By virtue of the deterministic theory of CDM, a general mathematical scheme of stochastic dynamic modelling for damage evolution has been proposed. The stochastic dynamic analysis of various types of damage, such as ductile, fatigue and creep damage, could be taken into account. This model is based on the potential of dissipation after introducing the randomized damage variable in a proper random field. The coupling among the randomized damage, stress and strain is automatically obtained through the stochastic damaged constitutive equations. Therefore, the stochastic dynamic analysis of structural damage could be integrated with the stress–strain analysis of structure using numerical methods, such as the finite element, boundary element and hybrid element methods.

In addition to the stochastic dynamic property of damage evolution, the proposed model can also describe some important properties embodied in the deterministic damage theory. Taking Lemaître theory as an example [see eqn (15)], the proposed model can at least include the following properties:

—The positive damage rate since the systematic term (drift coefficient) in our model is positive and controls the process of damage evolution.
—The effect of stress triaxiality on damage can be modelled because the variable Y in the systematic term contains the triaxiality ratio (σ_H/σ_{eq}).
—Damage under random loading is modelled through the variation of stress in both the systematic and fluctuation (diffusion) terms.
—The non-linearity of damage with regard to stress by its dependence on $\dot{p}$ and $\dot{\pi}$ which are non-linear functions of stress.
—The non-linearity of damage with regard to time or number of cycles by the term $(1-D_t)^{\alpha_0}$ corresponds to a non-linear stochastic differential equation in damage.

Finally, it should be noted that the proposed stochastic dynamic model will degenerate into the deterministic constitutive equations of CDM while the fluctuation term vanishes. Therefore, the proposed model can not only take into account the physical behaviour, but it also enables the damage analysis to be carried out simultaneously based on the deterministic theory and the stochastic dynamic theory.

REFERENCES

Arnold, L. (1974). *Stochastic Differential Equations: Theory and Applications*. John Wiley & Sons, New York.

Chaboche, J. L. (1977). Sur l'utilisation des variables d'état interne pour la description du comportement viscoplastique et de la rupture par endommagement. Symposium Franco-Polonais "*Problèmes non linéaires de Mécanique, Cracovie* 1977", pp. 137–159.

Chaboche, J. L. (1981). Continuous damage mechanics—a tool to describe phenomena before crack initiation. *Nucl. Engng Des.* **64**, 233–247.

Chaboche, J. L. (1988a). Continuum damage mechanics: part 1—general concepts. *J. Appl. Mech.* **55**, 59–64.

Chaboche, J. L. (1988b). Continuum damage mechanics: part 2—damage growth, crack initiation, and crack growth. *J. Appl. Mech.* **55**, 65–72.

Chow, C. L. and Wang, J. (1987a). An anisotropic theory of elasticity for continuum damage mechanics. *Int. J. Fract.* **33**, 3–16.

Chow, C. L. and Wang, J. (1987b). An anisotropic theory of continuum damage mechanics for ductile fracture. *Engng Fract. Mech.* **27**, 547–558.

Cordebois, J. P. and Sidoroff, F. (1982). Endommagement anisotrope en elasticite et plasticite. *J. Mech. Theor. Appl.*, Numero Special, 45.

Howard, R. A. (1971). *Dynamic Probabilistic Systems*, Vol. I, II. John Wiley & Sons, New York.

Itô, K. (1951). On stochastic differential equations. *Amer. Math. Soc. New York* (*Memoirs, Amer. Math. Soc.* 4).

Janson, J. (1977). Dugdale-crack in a material with continuous damage formation. *Engng Fract. Mech.* **9**, 891–899.

Kachanov, L. M. (1958). On the time to failure during creep. *Izv. AN SSSR, OTN*, No. 8, 26–31.

Kachanov, L. M. (1986). *Introduction to Continuum Damage Mechanics*. Martinus Nijhoff, Boston.

Kestin, J. and Bataille, J. (1977). Irreversible thermodynamics of continua and internal variables. In *Continuum Models of Discrete Systems* (Edited by J. W. Provan). Univ. of Waterloo Press.

Krajcinovic, D. (1984). Continuum damage mechanics. *Appl. Mech. Rev.* **37**(1), 1–6.

Krajcinovic, D. and Lemaître, J. (1987). Continuum damage mechanics theory and applications. Int. Center for Mech. Sci. Courses and Lecture, No. 295, by CSIM, Udine.

Ladeveze, P. and Lemaître, J. (1984). Damage effective stress in quasi-unilateral condition. IUTAM Congress, Lyngby, Denmark.

Lemaître, J. (1971). Evaluation of dissipation and damage in quasi-unilateral condition. Proc. ICM1, Tokyo.

Lemaître, J. (1984). How to use damage mechanics. *Nucl. Engng Des.* **80**, 233–245.

Lemaître, J. (1986). Local approach of fracture. *Engng Fract. Mech.* **25**(5/6), 523–537.

Lemaître, J. (1987). Formulation and identification of damage kinetic constitutive equations. In *Continuum Damage Mechanics Theory and Applications*, Springer, Wien, NY. (Edited by D. Krajcinovic and J. Lemaître), pp. 37–90.

Lemaitre, J. and Chaboche, J. L. (1978). Aspect phénoménologique de la rupture par endommagement. *J. Mécanique Appliquée* **2**(3), 317–365.

Lemaitre, J. *et al.* (1987). Damage measurements. *Engng Fract. Mech.* **28**(5/6), 643–661.

Lu, T. J. and Chow, C. L. (1990). On constitutive equations of inelastic solids with anisotropic damage. *Theor. Appl. Fract. Mech.* **14**, 187–218.

Murakami, S. (1987). Anisotropic aspects of material damage and application of continuum damage mechanics. In Continuum Damage Mechanics Theory and Applications (Edited by D. Krajcinovic and J. Lemaitre), 91–133.

Rabotnov, Yu. N. (1963). On the equations of state for creep. *Prog. Appl. Mech.* 307–315.

Rabotnov, Yu. N. (1969). Creep rupture. *Proc. XII Int. Congress Appl. Mech., Stanford.* Springer, Berlin.

Sidoroff, F. (1981). Description of anisotropic damage application to elasticity. In *IUTAM Colloquium on Physical Nonlinearities in Structural Analysis*, p. 237. Springer, Berlin.

Simo, J. C. and Ju, J. W. (1987a). Stress and strain based continuum damage models—I. Formulation. *Int. J. Solids Structures* **23**, 821–840.

Simo, J. C. and Ju, J. W. (1987b). Stress and strain based continuum damage models—II. Computational aspects. *Int. J. Solids Structures* **23**, 841–869.

Stratonovich, R. L. (1966). A new representation for stochastic integrals and equations. *SIAM J. Control* **4**, 362–371.